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Lower bounds for projective designs, cubature formulas and related isometric embeddings

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ABSTRACT

Yudin's lower bound [V.A. Yudin, Lower bounds for spherical designs, Izv. Math. 61 (3) (1997) 673–683] for the spherical designs is generalized to the cubature formulas on the projective spaces over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and thus to isometric embeddings $l^m_{2;\mathbb{K}} \to l^n_{p;\mathbb{K}}$ with $p \in 2\mathbb{N}$. For large p and in some other situations this result is substantially better than known before.

1. Introduction

In the theory of spherical and projective designs some important lower bounds were obtained [2, 8] by maximization of the functional

$$D(f) = \frac{f(1)}{c_0[f]}, \quad f \in K_l,$$

where K_l is the set of nonnegative on $(-1, 1)$ nonzero polynomials f, $\deg f \le l$, and

$$c_0[f] = \int_{-1}^{1} f(t)\omega_{\alpha,\beta}(t)\mathrm{d}t, \quad \omega_{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta. \tag{1.1}$$

Here the number l and the exponents $\alpha, \beta > -1$ are respectively depending on a design (actually, on its strength) and on its underlying space.

Obviously, $\sup\{D(f) : f \in K_l\} = \sup\{f(1) : f \in K_l, c_0[f] = 1\}$. The solution to the latter linear programming problem is classical, the extremal polynomial $f_{\max}$ is unique and can be expressed in terms of the Jacobi polynomials, see [27], Section 7.7.1. For the designs of cardinality n this yields

$$n \ge \tau_{\alpha,\beta} f_{\max}(1) = \tau_{\alpha,\beta} \max\{D(f) : f \in K_l\}, \tag{1.2}$$

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where

$$\tau_{\alpha,\beta} = \int_{-1}^{1} \omega_{\alpha,\beta}(t)\mathrm{d}t. \tag{1.3}$$

The purely combinatorial counterparts of the linear programming bound (1.2) are known since Delsart's seminal work [7]. Furthermore, such a bound turns out to be valid for the designs (weighted, in general) on polynomial spaces [10,17,18].

We denote by $L_2^{(\alpha,\beta)}(-1,1)$ the space of complex-valued measurable functions f on $(-1,1)$ such that

$$\|f\|^2 \equiv \int_{-1}^{1} |f(t)|^2 \omega_{\alpha,\beta}(t)\mathrm{d}t < \infty.$$

The corresponding Jacobi polynomials $P_k(t)$ constitute an orthogonal basis in $L_2^{(\alpha,\beta)}$, so that

$$f(t) = \sum_{k=0}^{\infty} \nu_k c_k[f] P_k(t), \quad f \in L_2^{(\alpha,\beta)}, \tag{1.4}$$

where

$$c_k[f] = \int_{-1}^{1} f(t) P_k(t) \omega_{\alpha,\beta}(t)\mathrm{d}t, \quad \nu_k = 1/\|P_k\|^2. \tag{1.5}$$

The Jacobi–Fourier series (1.4) converges to f in $L_2^{(\alpha,\beta)}(-1,1)$. The coefficient $c_0[f]$ in (1.5) coincides with that of (1.1) since $P_0(t) \equiv 1$, according to the usual standardization

$$\deg P_k = k, \qquad P_k(1) = \binom{\alpha+k}{k} \quad (k = 0, 1, 2, \ldots).$$

For the same reason $\nu_0 = 1/\tau_{\alpha,\beta}$.

The linear programming bound (1.2) can be extended to the set $K_{l,l'}$, $l' > l$, of the polynomials $f \neq 0$, $\deg f \leq l'$, such that $f(t) \geq 0$ for $|t| \leq 1$ and $c_k[f] \leq 0$ for $l+1 \leq k \leq l'$ (cf. [8], Theorem 5.10). On this base a series of new concrete lower bounds for the projective designs was obtained in [3–5]. For the spherical designs Yudin [29] considered the limit case $l' = \infty$. His class $K_{l,\infty}$ consists of all nonnegative nonzero continuous functions $f(t)$, $|t| \leq 1$, such that $c_k[f] \leq 0$ for all $k \geq l+1$. A suitable choice of a function $f \in K_{l,\infty}$ yields a lower bound asymptotically better than classical one that comes from (1.2).

In the present paper we generalize Yudin's result on the weighted designs (the cubature formulas) on the projective spaces $\mathbb{K}P^{m-1}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. (Recall that $\mathbb{H}$ is the standard notation for the quaternion field.) The extension to the cubature formulas is important, in particular, because of their equivalence to the isometric embeddings $l_{2;\mathbb{K}}^m \to l_{p;\mathbb{K}}^n$, $p \in 2\mathbb{N}$ [15,22,24,26]. Note that with the standard inner product (x,y) the space $l_{2;\mathbb{K}}^m$ is Euclidean, its unit sphere is $S = S^{\delta m-1}$, where $\delta = \delta(\mathbb{K})$ is the real dimension of $\mathbb{K}$, i.e $\delta(\mathbb{R}) = 1$, $\delta(\mathbb{C}) = 2$, $\delta(\mathbb{H}) = 4$. (In contrast, with $p \neq 2$ the space $l_{p;\mathbb{K}}^n$ is not Euclidean, moreover, it does not contain any m-dimensional Euclidean subspace, $m \geq 2$, as long as $p \notin 2\mathbb{N}$ [20,22].)

From now on we assume $m \geq 2$, $p \in 2\mathbb{N}$, and denote by $\Phi_{\mathbb{K}}(m,p)$ the space of complex-valued functions $\phi(x)$, $x \in S$, satisfying the following conditions, see [22,23].

(i) $\phi = \psi|S$, where ψ is a homogeneous polynomial of degree p ("p-forms") on the space $\mathbb{K}^m \equiv \mathbb{R}^{\delta m}$;
(ii) ψ is invariant in the sense that

$$\psi(w\alpha) = \psi(w) \quad (w \in \mathbb{K}^m, \alpha \in \mathbb{K}, |\alpha| = 1).$$

A fortiori, $\phi(x\alpha) = \phi(x)$ that allows us to naturally transfer ϕ to the projective space $\mathbb{K}P^{m-1}$. However, we will consider ϕ on S which is equivalent but more elementary. In this setting a projective *cubature formula of index p on S* is

$$\int \phi \mathrm{d}\sigma = \sum_{i=1}^{n} \phi(x_i)\rho_i, \quad \phi \in \Phi_{\mathbb{K}}(m,p), \tag{1.6}$$

where σ is the normalized Lebesgue measure on S, the nodes $x_i \in S$ are projectively distinct, and the weights ρ_i are positive. (Note that $\sum \rho_i = 1$ automatically by the restriction of $(x,x)^{p/2}$ to S.) In the case of equal ρ_i the set $\{x_i\}_1^n$ is nothing but a projective $p/2$-design, cf. [13].

2. Preliminary information

First of all, we have the decomposition

$$\Phi_{\mathbb{K}}(m,p) = \sum_{k=0}^{p/2} \mathrm{Harm}_{\mathbb{K}}(m,2k) \tag{2.1}$$

where the space $\mathrm{Harm}_{\mathbb{K}}(m,2k)$ consists of restrictions to S of the invariant harmonic $2k$-forms. Regarding to the inner product

$$(\psi_1,\psi_2) = \int \bar{\psi}_1\psi_2 \mathrm{d}\sigma$$

the decomposition (2.1) is orthogonal.

For any orthonormal basis $\{\phi_{ks}\}_{i=1}^{d_{m,2k}}$ of $\mathrm{Harm}_{\mathbb{K}}(m,2k)$ the *addition formula*

$$\sum_{s=1}^{d_{m,2k}} \overline{\phi_{ks}(x)}\phi_{ks}(y) = b_{m,k}P_k(xy) \quad (x,y \in S) \tag{2.2}$$

holds with

$$b_{m,k} = \tau_{\alpha,\beta}\nu_k P_k(1), \qquad \alpha = \frac{\delta(m-1)-2}{2}, \qquad \beta = \frac{\delta-2}{2}, \tag{2.3}$$

and

$$xy = 2|(x,y)|^2 - 1, \tag{2.4}$$

according to [22], cf.[12,16,25]. Later on we operate only with α,β given by (2.3).

Now let X be a finite nonempty subset of S, and let $A(X)$ be its angle set,i.e.

$$A(X) = \{xy : x,y \in X, x \neq y\}.$$

The addition formula easily implies the following important lemma, cf. [8,14,17,29] for $\lambda(x) \equiv 1$ and [9,18,22] for arbitrary $\lambda(x) > 0, \sum \lambda(x) = 1$. In all the quoted papers, except for [29], the function f is a polynomial.

Lemma 2.1. *Let the series*

$$\sum_{k=0}^{\infty} a_k P_k(t)$$

converge to a function $f(t)$ for every $t \in A(X)$ and for $t = 1$. Then

$$\sum_{x,y\in X} f(xy)\bar{\lambda}(x)\lambda(y) = \sum_{k=0}^{\infty} a_k b_{m,k}^{-1} \sum_{s=1}^{d_{m,2k}} \left| \sum_{x\in X} \phi_{ks}(x)\lambda(x) \right|^2, \tag{2.5}$$

where λ is an arbitrary function $X \to \mathbb{C}$.

The next lemma allows us to make use of Lemma 2.1.

Lemma 2.2. *The Jacobi–Fourier series of any function $f \in K_{l,\infty}$ converges to $f(t)$ for all $t \in [-1, 1]$.*

Proof. Since $f(t)$ is continuous, its Jacobi–Fourier series at $t = 1$ is summable to $f(1)$ by a Cesaro method, see [27], Theorem 9.1.3. Therefore, it is summable to $f(1)$ by the Abel method, see [11], Theorem 55. Hence, this series converges to $f(1)$ since $c_k[f] \le 0$ for $k \ge l+1$. It remains to refer to Theorem 7.32.1 from [27] which states that

$$\max_{|t|\le 1} |P_k(t)| = P_k(1) \tag{2.6}$$

if $\max(\alpha, \beta) \ge -1/2$. The latter is fulfilled because of (2.3) and $m \ge 2$. □

Corollary 2.3. *Formula (2.5) is true for every $f \in K_{l,\infty}$ with $a_k = \nu_k c_k[f]$, $k \ge 0$.*

Remark 2.4. In [29] the absolute convergence of the corresponding series is mentioned without proof. The proof of Lemma 2.2 shows that in our situation the convergence is indeed absolute and uniform.

Now we prove the linear programming bound that we need, cf. [9,18,22,29].

Proposition 2.5. *The inequality*

$$n \ge \tau_{\alpha,\beta} \sup \left\{D(f) : f \in K_{p/2,\infty}\right\} \tag{2.7}$$

holds for any projective cubature formula of shape (1.6).

Proof. From (1.6) it follows that

$$\sum_{i=1}^{n} \phi(x_i)\rho_i = 0, \quad \phi \in \mathrm{Harm}_{\mathbb{K}}(m, 2k), \quad 1 \le k \le p/2. \tag{2.8}$$

Applying Corollary 2.3 and Lemma 2.1 to $f \in K_{p/2,\infty}$, $X = \{x_i\}_1^n$ and $\lambda(x) = \rho(x)$, $x \in X$, we obtain

$$f(1) \sum_{x\in X} \rho^2(x) \le \sum_{x,y\in X} f(xy)\rho(x)\rho(y) \le a_0 b_{m,0}^{-1} \left(\sum \rho(x)\right)^2 .$$

Indeed, on the left-hand side of (2.5) all summands are ≥ 0. On the right-hand side the summands are ≤ 0 for $k \ge p/2+1$ and vanish for $1 \le k \le p/2$ by (2.8). It remains to recall that $\sum \rho(x) = 1$, therefore, $\sum \rho^2(x) \ge n^{-1}$; on the other hand, $a_0 b_{m,0}^{-1} = c_0[f]/\tau_{\alpha,\beta}$ since $b_{m,0} = 1$.

Remark 2.6. The inequality (2.7) implies

$$n \ge \tau_{\alpha,\beta} \sup\{D(f) : f \in K_{p/2,l'}\}, \quad l' \ge p/2,$$

since $K_{p/2,l'} \subset K_{p/2,\infty}$. (For $l' = l$ we set $K_{l',l} = K_l$.)

Since with any given m, p a projective cubature formula exists (or, equivalently, there exists an isometric embedding $l_{2;\mathbb{K}}^m \to l_{p;\mathbb{K}}^n$), we have

Corollary 2.7. $\sup\{D(f) : f \in K_{p/2,\infty}\} < \infty$.

The supremum in question is unknown but a "good" test function can be constructed using the "convolution"

$$\int g(xu)h(uy)\mathrm{d}\sigma(u) \quad (x, y \in S) \tag{2.9}$$

of two suitable functions $g(t)$ and $h(t)$, $-1 \le t \le 1$, cf. [29].

Lemma 2.8. *For any $e \in L_1^{(\alpha,\beta)}(-1, 1)$ the function $u \mapsto e(xu)$, $u \in S$, belongs to $L_1(S, \sigma)$ for every $x \in S$ and*

$$\int e(xu)\mathrm{d}\sigma(u) = \frac{1}{\tau_{\alpha,\beta}} \int_{-1}^{1} e(t)\omega_{\alpha,\beta}(t)\mathrm{d}t. \tag{2.10}$$

Proof. This follows by calculation in the spherical coordinates consistently introduced in $\mathbb{R}^\delta$ and $\mathbb{R}^{\delta(m-1)}$.

Corollary 2.9. *With $g, h \in L_2^{(\alpha,\beta)}(-1, 1)$ the integral (2.9) exists for all $x, y \in S$.*

Since any ordered pair $x', y' \in S$ with $x'y' = xy$ can be obtained from x, y by an isometry of $l^m_{2;\mathbb{K}}$, the integral (2.9) depends on xy only. Thus, we have a function $(g * h)(t)$, $-1 \le t \le 1$, such that

$$(g * h)(xy) = \int g(xu)h(uy)\mathrm{d}\sigma(u). \tag{2.11}$$

In particular, for $x = y$ (2.11) yields

$$(g * h)(1) = \int g(xu)h(xu)\mathrm{d}\sigma(u) = \frac{1}{\tau_{\alpha,\beta}} \int_{-1}^{1} g(t)h(t)\omega_{\alpha,\beta}(t)\mathrm{d}t, \tag{2.12}$$

by (2.10). Moreover, applying the Schwartz inequality to (2.11) and using (2.10) again we obtain

$$\sup_t |(g * h)(t)| \le \frac{1}{\tau_{\alpha,\beta}} \|g\| \cdot \|h\|. \tag{2.13}$$

By this inequality and bilinearity, the convolution $g * h$ determines a continuous mapping $(L_2^{(\alpha,\beta)})^2 \to L_\infty$.

Lemma 2.10. *With $g, h \in L_2^{(\alpha,\beta)}$ the function $(g * h)(t)$ is continuous, and the series*

$$(g * h)(t) = \sum_{k=0}^{\infty} (\nu_k^2/b_{m,k})c_k[g]c_k[h]P_k(t)$$

converges uniformly.

Proof. Let

$$g_N = \sum_{j=0}^{N} \nu_j c_j[g]P_j, \quad h_N(t) = \sum_{k=0}^{N} \nu_k c_k[h]P_k.$$

Then

$$g_N * h_N = \sum_{k=0}^{N} (\nu_k^2/b_{m,k})c_k[g]c_k[h]P_k$$

since

$$P_j * P_k = b_{m,k}^{-1} P_k \delta_{jk}$$

by the addition formula. Since $g_N \to g$ and $h_N \to h (N \to \infty)$ in $L_2^{(\alpha,\beta)}$, we obtain $g_N * h_N \to g * h$ uniformly. Thus, the limit function is continuous. □

Corollary 2.11. $c_k[g * h] = \nu_k c_k[g]c_k[h]/b_{m,k} = c_k[g]c_k[h]/\tau_{\alpha,\beta}P_k(1)$.

3. A function $f_l \in K_{l,\infty}$

Recall that all roots of every P_k are simple and lie on $(-1, 1)$. The roots of the derivative P_k' alternate them, so they are also simple and lie on $(-1, 1)$. Now we introduce a function f_l by setting

$$f_l = g * h, \qquad g(t) = \begin{cases} P_r(t) - P_r(\xi), & t \geq \xi \\ 0, & t < \xi, \end{cases} \qquad h(t) = \begin{cases} 1, & t \geq \xi \\ 0, & t < \xi, \end{cases} \tag{3.1}$$

where $r = l + 1$ and ξ is the largest root of P_r', cf. [29]. We have to verify that $f_l \in K_{l,\infty}$.

By Lemma 2.10 f_l is continuous. The inequality $f_l \geq 0$ follows from (2.11) since $h \geq 0$ and $g \geq 0$. The former is obvious, the latter is true since $g(\xi) = 0$, $g(1) \geq 0$ and $g'(t) \neq 0$ for $\xi < t \leq 1$. Moreover, $f_l(1) > 0$ by (2.12), thus, $f_l \neq 0$. It remains to prove that $c_k[f_l] \leq 0$ for $k \geq r$. In [29] a rather complicated vector analysis on $\mathbb{R}^m$ was used at this point. We manage without a generalization of this technique to $\mathbb{C}^m$ and $\mathbb{H}^m$ by dealing with the corresponding Jacobi polynomials immediately. Our approach is also applicable to the octonian projective line and plane, in spite of absence vector spaces over $\mathbb{O}$ because of nonassociativity. To this end it suffices to set $(\alpha, \beta) = (3, 3)$ for $\mathbb{O}P^1$ and $(7, 3)$ for $\mathbb{O}P^2$ in what follows.

Our starting point is the differential equation

$$\Delta_i \equiv (\omega_{\alpha+1,\beta+1} P_i')' + i(i + \lambda)\omega_{\alpha,\beta} P_i = 0, \quad i \geq 0, \tag{3.2}$$

where $\lambda = \alpha + \beta + 1$, see [27], formula (4.2.1). Note that $\lambda \geq 0$ by (2.3). From (3.2) it follows that

$$\begin{aligned} 0 &= \int_\xi^1 (P_r \Delta_k - P_k \Delta_r)\mathrm{d}t \\ &= (k - r)(k + r + \lambda) \int_\xi^1 \omega_{\alpha,\beta} P_r P_k \mathrm{d}t + \int_\xi^1 \{\omega_{\alpha+1,\beta+1}(P_r P_k' - P_k P_r')\}'\mathrm{d}t \\ &= (k - r)(k + r + \lambda) \int_\xi^1 \omega_{\alpha,\beta} P_r P_k \mathrm{d}t - (\omega_{\alpha+1,\beta+1} P_r P_k')(\xi) \end{aligned}$$

since $\omega_{\alpha+1,\beta+1}(1) = 0$, $P_r'(\xi) = 0$. For $k \neq r$ we obtain

$$\int_\xi^1 P_r P_k \omega_{\alpha,\beta} \mathrm{d}t = \frac{(\omega_{\alpha+1,\beta+1} P_r P_k')(\xi)}{(k - r)(k + r + \lambda)}.$$

This formula extends to $r = 0$ since $P_0(t) \equiv 1$, so $P_0'(\xi) = 0$. Thus,

$$\int_\xi^1 P_k \omega_{\alpha,\beta} \mathrm{d}t = \frac{(\omega_{\alpha+1,\beta+1} P_k')(\xi)}{k(k + \lambda)},$$

and then

$$\int_\xi^1 P_r P_k \omega_{\alpha,\beta} \mathrm{d}t = \frac{k(k + \lambda) P_r(\xi)}{(k - r)(k + r + \lambda)} \int_\xi^1 P_k \omega_{\alpha,\beta} \mathrm{d}t.$$

As a result,

$$c_k[g] = \int_\xi^1 g P_k \omega_{\alpha,\beta} \mathrm{d}t = \frac{r(r + \lambda) P_r(\xi)}{(k - r)(k + r + \lambda)} \int_\xi^1 P_k \omega_{\alpha,\beta} \mathrm{d}t = \frac{r(r + \lambda) P_r(\xi)}{(k - r)(k + r + \lambda)} c_k[h],$$

and, by Corollary 2.11,

$$c_k[f_l] = \frac{r(r + \lambda) P_r(\xi)}{(k - r)(k + r + \lambda) P_k(1) \tau_{\alpha,\beta}} (c_k[h])^2 \quad (k \neq r). \tag{3.3}$$

Since $P_k(1) > 0$, formula (3.3) yields $\operatorname{sign} c_k[f_l] = \operatorname{sign}(P_r(\xi))$, $\quad k > r$. But $\operatorname{sign} P_r(\xi) = -1$ since ξ lies in between two largest roots of $P_r(t)$ and $P_r(1) > 0$. Thus, $c_k[f_l] < 0$ for $k > r$. In addition, $c_r[f_l] = 0$ since $c_r[h] = 0$. The latter follows from (3.2) with $i = r$ by integration over $[\xi, 1]$.

In conclusion we note that ξ in (3.1) is actually the largest root of $P_l^{(\alpha+1,\beta+1)}(t)$, see [27], formula (4.21.7).

4. Main theorem

Now we are in position to prove the following.

Theorem 4.1. *The number n of nodes of every projective cubature formula of index p on $S^{\delta m-1}$ satisfies the inequality*

$$n \geq \frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)F(-\beta,\alpha+1,\alpha+2,\varepsilon)}\left(\frac{1}{\varepsilon}\right)^{\delta(m-1)/2}, \tag{4.1}$$

where F is the hypergeometric function, the numbers α and β are given by (2.3), $\varepsilon = (1-\xi)/2$, ξ is the largest root of the Jacobi polynomial $P_{p/2}^{(\alpha+1,\beta+1)}(t)$.

Proof. Using $f_{p/2}(t)$ as a test function in (2.7) we get

$$n \geq \frac{\tau_{\alpha,\beta} f_{p/2}(1)}{c_0[f_{p/2}]}.$$

By (2.12) and (3.1) we have

$$\tau_{\alpha,\beta} f_{p/2}(1) = \int_{\xi}^{1} gh\omega_{\alpha,\beta}\mathrm{d}t = \int_{\xi}^{1} g\omega_{\alpha,\beta}\mathrm{d}t = c_0[g].$$

On the other hand, $c_0[f_{p/2}] = c_0[g]c_0[h]/\tau_{\alpha,\beta}$ by Corollary 2.11. Hence,

$$n \geq \frac{\tau_{\alpha,\beta}}{c_0[h]} = \frac{\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\mathrm{d}t}{\int_{\xi}^{1}(1-t)^{\alpha}(1+t)^{\beta}\mathrm{d}t}. \tag{4.2}$$

Now we substitute $t = 1-2s$ into the numerator and $t = 1-2\varepsilon s$ into the denominator. This yields (4.1) since

$$F(-\beta,\alpha+1,\alpha+2,\varepsilon) = (\alpha+1)\int_{0}^{1} s^{\alpha}(1-\varepsilon s)^{\beta}\mathrm{d}s \tag{4.3}$$

(cf. [1], formula (15.3.1)) and

$$\int_{0}^{1} s^{\alpha}(1-s)^{\beta}\mathrm{d}s = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

(Also note that $\alpha+1 = (\delta m-\delta)/2$ by (2.3).) □

Remark 4.2. By the substitution $t = 2s^2-1$ in (4.2) we obtain

$$n \geq \frac{\int_{0}^{1}(1-s^2)^{\alpha}s^{2\beta+1}\mathrm{d}s}{\int_{\eta}^{1}(1-s^2)^{\alpha}s^{2\beta+1}\mathrm{d}s}, \quad \eta = \sqrt{(1+\xi)/2}.$$

In particular, for $\mathbb{K} = \mathbb{R}$ we have $\alpha = (m-3)/2$, $\beta = -1/2$, see (2.3). Hence,

$$n \geq \frac{\int_{0}^{1}(1-s^2)^{(m-3)/2}\mathrm{d}s}{\int_{\eta}^{1}(1-s^2)^{(m-3)/2}\mathrm{d}s} \quad (\mathbb{K} = \mathbb{R}), \tag{4.4}$$

where η is the largest root of the polynomial

$$P_{p/2}^{((m-1)/2,1/2)}(2s^2-1) = \mathrm{const}\cdot P_{p+1}^{((m-1)/2,(m-1)/2)}(s)/s, \tag{4.5}$$

or, equivalently, of the Gegenbauer polynomial $C_{p+1}^{m/2}(s)$ (see [27], formulas (4.1.5) and (4.7.1)). In the case of antipodal spherical $(p+1)$-design the lower bound (4.4) turns into (3) of [29] up to the factor 2 in the latter. This factor is nothing but the degree of the natural covering $S^{m-1} \to \mathbb{R}P^{m-1}$.

Remark 4.3. By (2.3) we have $\alpha = m-2$, $\beta = 0$ for $\mathbb{K} = \mathbb{C}$, and $\alpha = 2m-3$, $\beta = 1$ for $\mathbb{K} = \mathbb{H}$. Accordingly, (4.1) yields

$$n \geq \left(\frac{1}{\varepsilon}\right)^{m-1} \quad (\mathbb{K} = \mathbb{C}) \tag{4.6}$$

and

$$n \geq \frac{1}{(2m-1)-(2m-2)\varepsilon}\left(\frac{1}{\varepsilon}\right)^{2m-2} \quad (\mathbb{K} = \mathbb{H}) \tag{4.7}$$

by (4.3). Also

$$n \geq \frac{1}{\Phi_m(\varepsilon)}\left(\frac{1}{\varepsilon}\right)^{4m-4} \quad (\mathbb{K} = \mathbb{O}), \tag{4.8}$$

where $m \in \{2, 3\}$ and

$$\Phi_2(\varepsilon) = 35 - 84\varepsilon + 70\varepsilon^2 - 20\varepsilon^3, \qquad \Phi_3(\varepsilon) = 165 - 440\varepsilon + 396\varepsilon^2 - 120\varepsilon^3.$$

In the real case the hypergeometric function in (4.1) is not a polynomial of ϵ.

Now we denote by $N_{\mathbb{K}}(m, p)$ the minimal number n of nodes in the cubature formula (1.6) or, equivalently, the minimal n such that there is an isometric embedding $l^m_{2;\mathbb{K}} \to l^n_{p;\mathbb{K}}$. In this notation Theorem 4.1 states that

$$N_{\mathbb{K}}(m, p) \geq \frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)F(-\beta, \alpha+1, \alpha+2, \varepsilon)}\left(\frac{1}{\varepsilon}\right)^{\delta(m-1)/2}. \tag{4.9}$$

We will compare this result to the linear programming bound (1.2) with $l = p/2$. An explicit form of the latter is

$$N_{\mathbb{K}}(m, p) \geq \Lambda_{\mathbb{K}}(m, q), \quad q = p/2, \tag{4.10}$$

where

$$\Lambda_{\mathbb{K}}(m, q) = \begin{cases} \dbinom{m+q-1}{m-1} & (\mathbb{K} = \mathbb{R}), \\ \dbinom{m+[q/2]-1}{m-1}\dbinom{m+[(q+1)/2]-1}{m-1} & (\mathbb{K} = \mathbb{C}), \\ \dfrac{1}{2m-1}\dbinom{2m+[q/2]-2}{2m-2}\dbinom{2m+[(q+1)/2]-1}{2m-2} & (\mathbb{K} = \mathbb{H}) \end{cases} \tag{4.11}$$

see [8,15,19,22,24,26]. By the way,

$$N_{\mathbb{K}}(m, p) \leq \Lambda_{\mathbb{K}}(m, p), \tag{4.12}$$

see [15,22,24,26].

5. Asymptotic analysis

Given two positive-valued functions $a(p)$ and $b(p)$ on a semiaxis $p > p_0$, we write $a(p) \gtrsim b(p)$ as $p \to \infty$, if $\liminf(a(p)/b(p)) \geq 1$. From (4.10) and (4.11) it follows that

$$N_{\mathbb{K}}(m, p) \gtrsim \frac{p^{\delta(m-1)}}{\lambda_{\mathbb{K}}(m)}, \quad p \to \infty, \tag{5.1}$$

where

$$\lambda_{\mathbb{K}}(m) = \begin{cases} 2^{m-1}(m-1)! & \mathbb{K} = \mathbb{R}, \\ 2^{4(m-1)}(m-1)!^2 & \mathbb{K} = \mathbb{C}, \\ 2^{8(m-1)}(2m-1)!(2m-2)!, & \mathbb{K} = \mathbb{H} \end{cases} \tag{5.2}$$

or, in an unified form,

$$\lambda_{\mathbb{K}}(m) = \frac{\Gamma(\delta m/2)\Gamma(\delta(m-1)/2+1)}{\Gamma(\delta/2)} \cdot 2^{2\delta(m-1)} = \frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+2)}{\Gamma(\beta+1)} \cdot 2^{2\delta(m-1)}. \tag{5.3}$$

As to (4.9), ε is the only parameter depending on p. (Of course, ε also depends on m.) By definition, $\varepsilon = (1-\xi)/2 = \sin^2(\theta/2)$ where $\theta = \arccos \xi$. This θ is the smallest root of the polynomial $P_{p/2}^{(\alpha+1,\beta+1)}(\cos\theta)$. By Theorem 8.1.2 from [27] we have $\theta \sim 2j_{\alpha+1,1}/p$ where $j_{\alpha+1,1}$ is the smallest positive root of the Bessel function $J_{\alpha+1}(z)$. Therefore, $\varepsilon \sim j_{\alpha+1,1}^2/p^2$, and (4.9) yields

$$N_{\mathbb{K}}(m,p) \gtrsim \frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \cdot \frac{p^{\delta(m-1)}}{j_{\alpha+1,1}^{\delta(m-1)}}, \quad p \to \infty, \tag{5.4}$$

since $\varepsilon \to 0$, $F(\cdot,\cdot,\cdot,0) = 1$. *This estimate is better than* (5.1) because of

Proposition 5.1. *The inequality*

$$j_{\alpha+1,1}^{\delta(m-1)} < \frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\lambda_{\mathbb{K}}(m) \tag{5.5}$$

holds for all $m \geq 2$, except for the case $m = 2$, $\delta = 1$, when (5.5) *changes to an equality.*

Proof. By (5.3) the inequality (5.5) is equivalent to

$$j_{\alpha+1,1}^{\delta(m-1)} < \Gamma(\alpha+2)^2 \cdot 2^{2\delta(m-1)}. \tag{5.6}$$

We set $\alpha + 1 = \nu$, so that $\delta(m-1) = 2\nu$, and (5.6) takes the form

$$j_{\nu,1}^{2\nu} < \Gamma(\nu+1)^2 \cdot 16^{\nu}. \tag{5.7}$$

The number ν is positive integer or half-integer, $\nu \geq 1/2$, and $\nu = 1/2$ if and only if $m = 2$, $\delta = 1$. In this case $j_{\nu,1} = \pi$ since $J_{1/2}(z)$ is proportional to $\sin z/\sqrt{z}$. On the other hand, $\Gamma(3/2)^2 \cdot 16^{1/2} = \pi$ as well. Thus, (5.7) changes to an equality.

Now let $\nu \geq 1$. By the inequality $j_{\nu,1} < \sqrt{2(\nu+1)(\nu+3)}$ (see [28], section 15.3) it suffices to prove that

$$(\nu+1)^{\nu}(\nu+3)^{\nu} \leq \Gamma(\nu+1)^2 \cdot 8^{\nu}. \tag{5.8}$$

By Stirling's lower bound the inequality (5.8) follows from

$$\left(1+\frac{1}{\nu}\right)^{\nu}\left(1+\frac{3}{\nu}\right)^{\nu} < 2\pi\nu\left(\frac{8}{e^2}\right)^{\nu}.$$

A fortiori, (5.8) follows from

$$2\pi\nu\left(\frac{8}{e^2}\right)^{\nu} > e^4.$$

But the latter is indeed true if $\nu \geq \nu_0$ where ν_0 is a unique root of the equation $2\pi\nu(8/e^2)^{\nu} = e^4$. It is easy to see that $\nu_0 < 6$, so (5.8) is valid for $\nu \geq 6$. For $\nu < 6$, i.e. $\nu = 1, 3/2, 2, \ldots, 5, 11/2$, the inequality (5.8) can be checked numerically. $\square$

The inequalities (5.1) and (5.4) can be rewritten as

$$\liminf_{p\to\infty} p^{-\delta(m-1)} N_{\mathbb{K}}(m,p) \geq 1/\lambda_{\mathbb{K}}(m) \tag{5.9}$$

and

$$\liminf_{p\to\infty} p^{-\delta(m-1)} N_{\mathbb{K}}(m,p) \geq \frac{\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} j_{\alpha+1,1}^{-\delta(m-1)} \tag{5.10}$$

respectively. By Proposition 5.1 the ratio $\kappa_{\mathbb{K}}(m)$ of the lower bounds (5.9) and (5.10) is less than 1, except for the case $m = 2$, $\delta = 1$. Moreover, $\kappa_{\mathbb{K}}(m)$ exponentially decays as $m \to \infty$. Indeed,

$$\kappa_{\mathbb{K}}(m) = \frac{j_{\nu,1}^{2\nu}}{\Gamma(\nu+1)^2 \cdot 16^{\nu}}. \tag{5.11}$$

As $\nu \to \infty$, the right-hand side of (5.11) is asymptotically equal to

$$\frac{1}{2\pi\nu}\left(\frac{e}{4}\right)^{2\nu} e^{c(8\nu)^{1/3}}, \quad c = 1.855\ldots, \tag{5.12}$$

see [19], p. 119, where the asymptotic expression (5.12) appears with $\nu = m/2$ in the context of spherical designs. (Recall that in our context $\nu = \delta(m-1)/2$.)

Remark 5.2. From (4.12) the asymptotic upper bound

$$\limsup_{p\to\infty} p^{-\delta(m-1)} N_{\mathbb{K}}(m,p) \leq 2^{\delta(m-1)}/\lambda_{\mathbb{K}}(m) \tag{5.13}$$

follows. We see that there is an exponential gap between (5.13) and (5.10) as $m \to \infty$. Indeed, the ratio of these bounds is $2^{\delta(m-1)}\kappa_{\mathbb{K}}(m)$. It is an open problem to reduce this gap as much as possible. To this end one can try to improve the upper bound (4.12). However, the only known reduction of that is by 1 [6,21].

6. The case $m = 2$

In this case we discuss the real, complex and quaternion situations separately.

6.1. $\mathbb{K} = \mathbb{R}$.

Then the inequalities (4.9) *and* (4.10) *are both the equalities, so they coincide.* Indeed, $N_{\mathbb{R}}(2,p) = p/2+1$, according to [8,24,26], and, on the other hand, $\Lambda_{\mathbb{R}}(2,q) = q+1 = p/2+1$ by (4.11). Furthermore, in the real case (4.9) is equivalent to (4.4). For $m = 2$ this yields $N_{\mathbb{R}}(2,p) \geq \pi/2 \arccos \eta = p/2+1$. Indeed, in this context η is the largest root of the Gegenbauer polynomial $C_{p+1}^{1}(s) = \sin(p+2)\theta/\sin\theta$ where $\theta = \arccos s$.

6.2. $\mathbb{K} = \mathbb{C}$.

By (4.11)

$$N_{\mathbb{C}}(2,p) \geq \left[\left(\frac{p}{4}+1\right)^2\right]. \tag{6.1}$$

On the other hand, our bound (4.6) for $m = 2$ is

$$N_{\mathbb{C}}(2,p) \geq \left]\frac{2}{1-\xi_p}\right[\tag{6.2}$$

Table 1

p	2	4	6	8	10	12	14	16	18	20
$\Delta_{\mathbb{H}}(p)$	0	−1	−1	−4	−2	−6	−3	−6	1	−1

Table 2

p	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
$\Delta_{\mathbb{H}}(p)$	12	14	35	42	75	90	138	165	231	274	364	426	544	631	782

where ξ_p is the largest root of $P_{p/2}^{(1,1)}(t)$ and $]\zeta[$ means the smallest integer $\geq \zeta$, $\zeta \in \mathbb{R}$. A numerical evaluation shows that (6.2) coincides with (6.1) for $p \leq 16$, but exceeds it for $18 \leq p \leq 90$. Moreover, the difference $\Delta_{\mathbb{C}}(p)$ between the lower bounds (6.2) and (6.1) is nondecreasing in this range, as we see from the table

p	≤16	18–24	26, 28	30, 32	34, 36	38	40	42, 44	46	48	50	52
$\Delta_{\mathbb{C}}(p)$	0	1	2	3	4	5	6	7	8	9	10	11

p	54	56	58	60	62	64	66	68	70	72	74	76	78	80	82	84	86	88	90
$\Delta_{\mathbb{C}}(p)$	12	13	14	15	17	18	19	20	22	23	25	26	28	29	31	32	34	36	38

The table also shows that the "derivative" $\Delta'_{\mathbb{C}}(p) = \Delta_{\mathbb{C}}(p) - \Delta_{\mathbb{C}}(p-2)$ is nondecreasing (rather slowly), so $\Delta_{\mathbb{C}}(p)$ is convex.

6.3. $\mathbb{K} = \mathbb{H}$

We have

$$N_{\mathbb{H}}(2, p) \geq \frac{1}{3}\binom{[p/2]+2}{2}\binom{[(p+2)/2]+3}{2} \tag{6.3}$$

from (4.10) and (4.11), but (4.7) yields

$$N_{\mathbb{H}}(2, p) \geq \left] \frac{4}{(2+\eta_p)(1-\eta_p)^2} \right[\tag{6.4}$$

where η_p is the largest root of $P_{p/2}^{(2,2)}(t)$.

Comparing (6.4) to (6.3) one can see a small advantage of (6.3) when $4 \leq p \leq 20$, $p \neq 18$. Namely, for the difference $\Delta_{\mathbb{H}}(p)$ between the lower bounds (6.4) and (6.3) we have Table 1.

However, for $p \geq 22$ this difference increases rather rapidly (see Table 2).

Also, an interesting observable phenomenon is a regular oscillation of $\Delta'_{\mathbb{H}}(p)$ in contrast to the monotonicity of $\Delta'_{\mathbb{C}}(p)$. Indeed, in both Tables 1 and 2 we have

$$\operatorname{sign} \Delta''_{\mathbb{H}}(p) = (-1)^{p/2+1} \tag{6.5}$$

for the second difference. This can be conjectured for all p as well as the results of the observations above.

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